Universal statistical properties of poker tournaments

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We present a simple model of Texas hold'em poker tournaments which contains the two main aspects of the game: *i.* the minimal bet is the blind, which grows exponentially with time; *ii.* players have a finite probability to go "all-in", hence betting all their chips. The distribution of the number of chips of players not yet eliminated (measured in units of its average) is found to be independent of time during most of the tournament, and reproduces accurately Internet poker tournaments data. This model makes the connection between poker tournaments and the persistence problem widely studied in physics, as well as some recent physical models of biological evolution or competing agents, and extreme value statistics which arises in many physical contexts.

Physicists are now more then ever involved in the study of complex systems which do not belong to the traditional realm of their science. Finance (options theory,...) [1], human networks (Internet, airports,...) [2], the dynamics of biological evolution [3, 4] and in general of competitive "agents" [5] are just a few examples of problems recently addressed by statistical physicists. However, many of these systems are not isolated and are thus sometimes very difficult to describe quantitatively: a financial model cannot predict the occurrence of wars or natural disasters which certainly affect financial markets, nor can it include the effect of all important external parameters (China's GDP growth, German exports...). Rather, these studies try to capture important qualitative features which, interestingly, are sometimes *universal*.

In this Letter, we study a very human and futile activity: poker tournaments. Although *a priori* governed by human laws (bluff, prudence, aggressiveness...), we shall find that some of their properties can be quantitatively described. One of the nice aspects of a poker tournament lies in the obvious fact that it is a truly *isolated system*. Two famous mathematicians [6, 7] contributed to the science of poker. However, they concentrated on finding the best strategies in head-to-head games (like most of their followers [8]). Here, we introduce a simple model which can be treated analytically and which reproduces some properties of Internet poker tournaments. Our main quantities of interest are the number of surviving players, the distribution of their chips amount ("stack"), the number of "chip leaders" during the tournament, and the distribution of their stack. Interestingly, the constraint that a surviving player must keep a positive stack relates poker tournaments to the problem of persistence [9, 10], and the competitive nature of the game connects some of our results with recent models of competing agents [3, 4, 5]. In addition, the properties of the "chip leader" display extreme value statistics, a phenomenon observed in many physical systems [12].

We now describe the main aspects of a Texas hold'em poker tournament (currently the most popular form of poker). Initially, N_0 players sit at tables accepting up to T players. In real poker tournaments, N_0 typically lies in the range $N_0 \sim 10 - 10000$, and the number of players per table is $T = 10$. We will not detail the precise rules of Texas hold'em poker, as we shall see that their actual form is totally irrelevant provided that two crucial ingredients of the game are kept:

• Before a game starts, the two players to the left of the dealer post "blind" bets. The blinds ensure that there is some money in the pot to play for at the very start of the game. The blind b *increases exponentially* with time, and typically changes to the value 40 \$, 60 \$, 100 \$, 150 \$, 200 \$, 300 \$, 400 \$,... every 10-15 minutes on Internet tournaments, hence being multiplied by a factor 10 every hour or so. We shall see that the growth rate of the blind entirely controls the pace of a tournament. *Therefore, the fact that the blind (which is also the minimal bet) grows exponentially with time must be a major ingredient of any realistic model of poker.*

• The next players post their bets ($\geq b$) according to their evaluation of the two cards they each receive. There are subsequent rounds of betting following the successive draws of five common cards. Ultimately, the betting player with the best hand (when combined with the common cards) wins the pot. Most of the deals end up with a player winning a small multiple of the blind. However, during certain deals, two or more players can aggressively raise each other, so that they finally bet a large fraction, if not all, of their chips. This can happen when a player goes "all-in", hence betting all his chips. *Any serious model of poker should take into account the fact that players often bet a few blinds, but sometimes end up betting all or a large fraction of their chips.*

Once a player loses all his chips, he is eliminated. During the course of the tournament, some players may be redistributed to other tables, in order to keep the number of tables minimum. Retaining the two main ingredients mentioned above, we now define a simple version of poker which turns out to describe *quantitatively the evolution of real poker tournaments*. The N_0 initial players (T players per table) receive the same amount of chips $x_0 \gg b_0$, where b_0 is the initial blind (x_0/b_0 is typically in the range $50 - 100$ in actual poker tournaments).

• The single "blinder" posts the blind, $b(t) =$

$b_0 \exp(t/t_0)$. For the following deal, the new blinder is the next player to the left of the current blinder.

• At each table (tables run in parallel), the players receive one card, c, which is a random number uniformly distributed between 0 and 1.

• The following players bet the value b with probability $e(c)$, if $0 \le c \le c_0$. $e(c)$ is an evaluation function, whose details will be immaterial. Intuitively, $e(c)$ should be an increasing function of c, implying that a player will more often play good hands than bad ones. We tried several forms of $e(c)$, obtaining the same results. In our simulations, we choose $e(c) = c^n$, where n is the number of players having already bet (including the blinder). In this case, $e(c)$ is simply the probability that c is the best card among $n+1$ random cards. This reflects the fact that a player should be careful when playing bad hands if many players have already bet. Determining the optimal evaluation function for a given T (in the spirit of Borel's and von Neumann's analysis for $T = 2$) is a formidable task which is left for a future study.

• The first player with a card $c > c_0$ goes all-in (hence $q = 1 - c_0$ is the probability to go all-in). The next players (including the blinder) can follow if their card is greater than c_0, and fold otherwise. If a player with a card $c > c_0$ cannot match the amount of chips of the first player all-in, he simply bets all his chips, but can only expect to win this amount from the other all-in players.

• Ultimately, the betting player with the highest card wins the pot (the blinder gets the blind back if nobody else bets). The players left with no chips are eliminated, and after each deal, certain players may be redistributed to other tables, in a process ensuring that the number of tables remains minimum at all times. After this round is completed at all tables, time is updated to $t+1$.

Let us first consider the unrealistic case $q = 0$. The amount of chips $x(t)$ of a given player evolves according to $x(t+1) = x(t) + \varepsilon(t)b(t)$, where $\varepsilon(t)$ has zero average (there is no winning strategy in the mathematical sense), and is Markovian, since successive deals are uncorrelated. If the typical value of $x \sim \bar{x}(t)$ remains significantly bigger than the blind $b(t)$, we can adopt a continuous time approach. Hence, the evolution of $x(t)$ is that of a generalized Brownian walker: $\frac{dx}{dt} = \sigma b(t)\eta(t)$, where $\sigma^2 = \bar{\varepsilon^2}$ is a constant of order unity, and $\eta(t)$ is a δ-correlated white noise. The number of surviving players with x chips, $P(x,t)$, evolves according to the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\sigma^2 b^2(t)}{2} \frac{\partial^2 P}{\partial x^2}, \qquad (1)$$

with the absorbing boundary condition $P(x = 0, t) = 0$, and initial condition $P(x, t = 0) = \delta(x - x_0)$. This kind of problem arises naturally in physics in the context of persistence, which is the probability that a random process $x(t)$ never falls below a certain level [9, 10]. Defining $\tau(t) = \sigma^2 b_0^2 t_0 (\mathrm{e}^{2t/t_0} - 1)/2$, Eq. (1) can be solved by the method of images [10]:

$$P(x,t) = \frac{N_0}{\sqrt{2\pi\tau(t)}} \left(\mathrm{e}^{-\frac{(x-x_0)^2}{2\tau(t)}} - \mathrm{e}^{-\frac{(x+x_0)^2}{2\tau(t)}} \right). \qquad (2)$$

For $t \gg t_0$, the chips distribution becomes scale invariant

$$P(x,t) = \frac{N(t)}{\bar{x}(t)} f\left(\frac{x}{\bar{x}(t)}\right), \qquad (3)$$

where the density of surviving players is given by

$$\frac{N(t)}{N_0} = \frac{2x_0}{\sqrt{\pi t_0}\sigma b_0} \mathrm{e}^{-\frac{t}{t_0}}. \qquad (4)$$

We find that the decay rate of the $N(t)$ is exactly the growth rate of the blind, which thus controls the pace of the tournament. The duration of a tournament t_{f} is

$$\frac{t_{\mathrm{f}}}{t_0} = \ln(N_0) - \frac{1}{2}\ln(t_0) + \ln\left(\frac{x_0}{b_0}\right), \qquad (5)$$

which only grows logarithmically with the number of players and the ratio x_0/b_0. The average stack is proportional to the blind $\bar{x}(t) = N_0 x_0/N(t) = \sqrt{\pi t_0}\sigma b(t)/2$. When $t_0 \gg 1$, this expression implies that $\bar{x}(t)/b(t) \gg 1$, hence validating the use of a continuous time approach. Finally, we find that the normalized distribution of chips is given by the Wigner distribution

$$f(X) = \frac{\pi}{2} X \mathrm{e}^{-\frac{\pi}{4}X^2}, \quad F(X) = 1 - \mathrm{e}^{-\frac{\pi}{4}X^2}, \qquad (6)$$

where $F(X) = \int_0^X f(Y)\, dY$. Equivalently, in the context of persistence, f is naturally found to be the first excited eigenstate of the quantum harmonic oscillator [10]. The scaling function f is *universal*, *i.e.* independent of all the microscopic parameters (b_0, t_0, x_0...). In the insert of Fig. 1, we plot the normalized distribution $f(X) = \bar{x}(t)P(x,t)/N(t)$ and $F(X)$ as a function of $X = x/\bar{x}(t)$, as obtained from extensive numerical simulations of the present poker model with $q = 0$. We find a perfect data collapse on the analytical result of Eq. (6).

We now consider the more realistic case $q > 0$. *A priori*, it seems that q is a new parameter whose precise value could dramatically affect the dynamics of the game. In reality, q must be intimately related to the decay rate t_0^{-1} of the number of players, which is imposed by the exponential growth of the blind. To see this, let us first compute the decay rate due to the all-in processes. At a given table, and for small q, the probability that an all-in process occurs is $P_{\mathrm{all-in}} = q^2 T(T-1)/2$, where the factor q^2 is the probability that two players go all-in, and $T(T-1)/2$ is the number of such pairs. Expecting $q \ll 1$, we have neglected all-in processes involving more than two players. During a two-player all-in process, there is a probability 1/2 that the losing player is the one with the smallest stack (he is then eliminated). Cumulating the results of the N/T tables, we find the density decay rate due to all-in processes

$$\frac{dN}{dt}_{\text{all-in}} = -\frac{1}{2}\times\frac{N}{T}\times P_{\text{all-in}} = -\frac{N}{t_{\text{all-in}}}, \quad (7)$$

$$t_{\text{all-in}} = \frac{4}{q^2(T-1)}. \quad (8)$$

We now make the claim that the physically optimal choice for $t_{\text{all-in}}$ (and hence for q) is such that the decay rate due to all-in processes is equal to the one caused by the chips fluctuations of order $b(t)$. Since the total decay rate should remain equal to t_0^{-1}, $t_{\text{all-in}} = 2t_0$ must hold (decay rates add up). If $t_{\text{all-in}} < 2t_0$, the game is dominated by all-in processes and $x(t)$ can get rapidly large compared to $b(t)$. The first player to go all-in is acting foolishly and takes the risk of being eliminated just to win the (negligible) blind. Inversely, if $t_{\text{all-in}} > 2t_0$, players (especially those with a declining stack) would be foolish not to make the most of the opportunity to double their stack by going all-in. We expect that real poker players would, on average, self-adjust their q to its optimal value. Finally, we find that q is not a free parameter, but should take the physical value

$$q = \sqrt{\frac{2}{(T-1)t_0}}. \quad (9)$$

We now write the exact evolution equation for the number of surviving players with x chips, combining the effect of pots of order b and all-in processes

$$\frac{\partial P}{\partial t} = \frac{\sigma^2 b^2}{2}\frac{\partial^2 P}{\partial x^2} + \frac{2}{t_0}(K(P) - P), \quad (10)$$

where the all-in kernel K is given by

$$K(P) = \frac{1}{4}P(x/2)\int_{x/2}^{+\infty}\frac{P(y)}{N}\,dy + \frac{1}{2}\int_0^{x/2}P(x-y)\frac{P(y)}{N}\,dy + \frac{1}{2}\int_0^{+\infty}P(x+y)\frac{P(y)}{N}\,dy, \quad (11)$$

and where we have dropped the time variable argument for clarity. In Eq. (10), the factor $\frac{2}{t_0} = q^2(T-1)$ is simply the rate of all-in processes involving the considered player, without presuming the outcome of the event. In addition, the first term of Eq. (11) describes processes where the considered player has doubled his stack by winning against a player with more chips than him. The second term corresponds to an all-in process where the player has won against a player with less chips than him (and has eliminated this player). Finally, the last term describes the loss against a player with less chips than him (otherwise the considered player is eliminated). Integrating Eq. (11) over x, we check that the probability to survive an all-in process is $\frac{3}{4}$, the two first terms adding up to $\frac{1}{2}$ (probability to win). We thus recover the decay rate associated to all-in processes, $\left(1-\frac{3}{4}\right)\times\frac{2}{t_0} = t_{\text{all-in}}^{-1}$.

We now look for a scaling solution of Eq. (11) of the form $P(x,t) = \lambda\hat{x}(t)^{-2}f\left(x/\hat{x}(t)\right)$, where the integral of f is normalized to 1, so that $N(t) = \lambda/\hat{x}(t)$. Plugging this *ansatz* into Eq. (10), we find that one must have $\hat{x}(t) \sim b(t)$ for all the terms to scale in the same manner. Defining $\hat{x}(t) = \sqrt{t_0}\sigma b(t)/2 \sim \bar{x}(t) \sim \mathrm{e}^{-t/t_0}$, and the scaling variable $X = x/\hat{x}(t)$, we find that f satisfies

$$f''(X) + Xf'(X) + \tfrac{1}{2}f(X/2)\int_{X/2}^{+\infty}f(Y)\,dY + \int_0^{X/2}f(X-Y)f(Y)\,dY + \tfrac{1}{2}\int_0^{+\infty}f(X+Y)f(Y)\,dY = 0, \quad (12)$$

with the boundary condition $f(0) = 0$. We did not succeed in solving this equation analytically. However, the small and large X behavior of $f(X)$ can be extracted from Eq. (12), $f(X) \sim_0 \frac{X}{2}$, and $f(X) \sim_{+\infty} 2\mu\mathrm{e}^{-\mu X}$. Thus, the universal scaling distribution decays more slowly than for $q = 0$. Eq. (12) can be solved numerically using a standard iteration scheme (∼2 minutes of CPU time on a PC workstation), and we find $\mu \approx 1.56$.

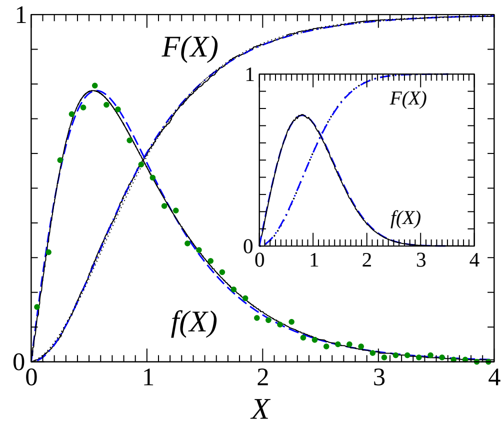

FIG. 1: We plot the normalized distribution of chips $f(X)$ and its cumulative sum $F(X)$ obtained from numerical simulations of our poker model (thin dotted lines, $N_0 = 10000$, $t_0 = 2000$, $x_0/b_0 = 100$, 10000 "tournaments" played). These distributions are extracted at times for which $N(t)/N_0 =$ 50%, 30%, 10% [Insert: for $q = 0$; dashed lines corresponds to the exact result of Eq. (6)]. The dashed lines correspond to the numerical solution of the exact Eq. (12). The data recorded from 20 real poker tournaments (totalizing 1584 players still in) are also plotted (full lines), and are found to agree remarkably with the present theory. Note that $f(X)$ for real tournaments was obtained by differentiating a fitting function to the actual cumulative sum. We also plot the standard but noisier bin plot of the distribution of chips in real poker tournaments (circles).

In Fig. 1, we plot the normalized distribution $f(X)$ as a function of $X = x/\bar{x}(t)$ obtained from extensive numerical simulations of the present poker model, with q given

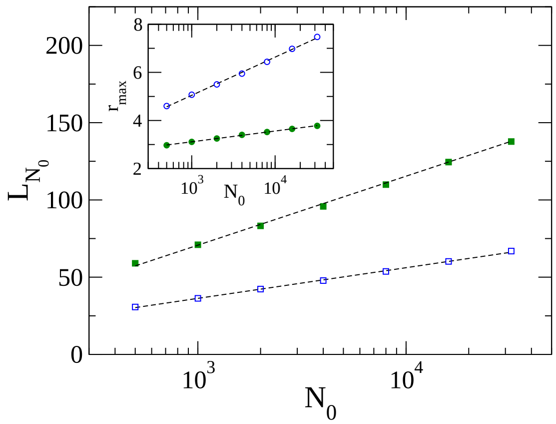

FIG. 2: We plot the average number of chip leaders L_{N_0} as a function of the number of initial players N_0, finding a convincing logarithmic growth (full symbols correspond to the case $q = 0$). The insert shows the logarithmic growth of $r_{\max}$ (defined in the text). The dashed lines correspond to log-linear fits of the data.

by Eq. (9). We find a perfect data collapse on the numerical solution of the exact scaling equation Eq. (12). In order to check the relevance of this parameter-free distribution to real poker tournaments, we visited two popular on-line poker playing zones, and followed 20 no-limit Texas hold'em tournaments with an initial number of players in the range $N_0 \sim 250-800$. When the number of players was down to the range $N \sim 60-130$, we manually recorded their number of chips [11]. Fig. 1 shows the remarkable agreement between these data and the results of the present model. The maximum of the distribution corresponds to players holding around 55% of the average stack. In addition, a player owning twice the average stack ($X = 2$) precedes 90% of the other players, whereas a player with half the average stack ($X = 1/2$) precedes only 25% of the other players.

We now consider the statistical properties of the player with the largest amount of chips at a given time, dubbed the "chip leader". First, we consider the average number of chip leaders L_{N_0} in a tournament with N_0 initial players. In many competitive situations arising for instance in biological evolution models [3, 4] or competing nodes in a driven network [5], it is found that L_N grows logarithmically with the number of competing agents N, a result which has been established analytically in a general competition model [4]. We confirm that in the present model, with or without all-in processes, the same phenomenon is observed (see Fig. 2). We have also computed the average maximum ratio $r_{\max} = \sup_t \bar{x}_{\rm lead}/\bar{x}$. In the model, $x_{\rm lead}/\bar{x}$ increases rapidly on a scale of order t_0, and then decays (almost linearly with time) to ~ 1.5, where it becomes non self-averaging due to large fluctuations at the end of the tournament. Fig. 2 illustrates the logarithmic growth of $r_{\max}$ as a function of N_0. For $N_0 = 500$, which is typical of Internet tournaments, we find $r_{\max} \approx 4.6$, which is fully compatible with real data.

Extreme value statistics have recently attracted a lot of attention from physicists in various contexts [12]. In this regard, we have checked that $z = (x_{\rm lead} - \bar{x}_{\rm lead})/(\bar{x^2}_{\rm lead} - \bar{x}^2_{\rm lead})^{1/2}$ is distributed according to the universal Gumbel distribution $g(z) = \pi \exp[-Z - \exp(-Z)]/\sqrt{6}$ (where $Z = \pi z/\sqrt{6} + \gamma$, and γ is Euler's constant), following a general property of independent or weakly correlated random variables [12, 13].

In this Letter, we have developed a quantitative theory of poker tournaments and made the connection between this problem and persistence in physics, the "leader problem" in evolution and competition models, and extreme value statistics. It would be interesting to obtain access to the full dynamical evolution of a large sample of real-life poker tournaments, in order to check the predictions of the model concerning the chip leader [11].

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